SEMESTER-4 (MTMG)

COURSE: CC/GE-4

RANDOM VARIABLE & DISTRIBUTION FUNCTION

1. Find the mean and variance of Poisson distribution with parameter λ .

Answer:
Let $\phi(t)$ be the m.g.f. of the Poisson distribution $P(\lambda)$.

Then $\phi(t)=E\left(e^{tx}\right)=\sum_{x=0}^{\infty}e^{tx}\frac{e^{-\lambda}\lambda^{x}}{x!}=e^{-\lambda}\sum_{x=0}^{\infty}\frac{\left(\lambda e^{t}\right)^{x}}{x!}=e^{-\lambda}\cdot e^{\lambda e^{t}}=e^{\lambda\left(e^{t}-1\right)}$

Now, $\quad \text{Mean}=E(X)=\phi'(0)$ $\qquad$ where $\phi'(t)=e^{\lambda\left(e^{t}-1\right)}\cdot\lambda e^{t}$

$=\lambda$

Again, $E\left(X^{2}\right)=\phi''(0)$ $\qquad$ where $\phi''(t)=\lambda^{2}e^{t}e^{\lambda\left(e^{t}-1\right)}+\lambda e^{t}e^{\lambda\left(e^{t}-1\right)}$

$=\lambda^{2}+\lambda$

Hence $\quad \text{Var}(X)=E\left(X^{2}\right)-\left\{E(X)\right\}^{2}=\lambda^{2}+\lambda-\lambda^{2}=\lambda$

Therefore, $\sigma_{X}=\sqrt{\lambda}$.

2. If the mean of a binomial distribution is 3 and the variance is $\frac{3}{2}$, find the probability of obtaining atmost 3 success.

Answer:

By the given condition $np=3,\ npq=\frac{3}{2}\ \therefore q=\frac{1}{2}$

Hence $\quad p=\frac{1}{2},n=6$

Let x denote the number of success

$\therefore P(x\le 3)=P(x=0)+P(x=1)+P(x=2)+P(x=3)$

$={}^{6}C_{0}\left(\frac{1}{2}\right)^{0}\left(\frac{1}{2}\right)^{6}+{}^{6}C_{1}\left(\frac{1}{2}\right)^{1}\left(\frac{1}{2}\right)^{5}+{}^{6}C_{2}\left(\frac{1}{2}\right)^{2}\left(\frac{1}{2}\right)^{4}+{}^{6}C_{3}\left(\frac{1}{2}\right)^{3}\left(\frac{1}{2}\right)^{3}$

$=\left(\frac{1}{2}\right)^{6}\left\{1+6+15+20\right\}=\frac{42}{64}=\frac{21}{32}$

3. A fair coin is tossed 400 times. Using normal approximation to binomial distribution find the probability of obtaining (i) exactly 200 heads (ii) between 190 and 210 heads, both inclusive. Given that the area under standard normal curve between z = 0 and z = 0.05 is 0.0199 and between z = 0 and z = 1.05 is 0.3531.

Answer:

Let X denote the no. of heads in 400 doses

Than $n=400,p=q=Y_{2},E(X)=np=200,\text{var}(X)=npq=100$

$P(X=200)=P\left(199.5<\hat{X}<200.5\right)$ $\qquad \therefore\sigma_{x}=10.$

$= P\left(\frac{199.5-200}{10} < z < \frac{200.5-200}{10}\right)$

$= P(-0.05 < z < 0.05)$

$= 2\times 0.0199 = 0.0398$

$P(190 < X < 210) = P\left(\frac{189.5-200}{10} < z < \frac{210.5-200}{10}\right)$

$= P(-1.05 < z < 1.05) = 2\times 0.3531 = 0.7062$

$P(X \ge 2) = P(X = 2) + P(X = 3) = 0.3 + 0.2 = 0.5$

$P(-2 < X \le 2) = P(X = -2) - P(X = 3) = 1 - 0.1 - 0.2 = 0.7$

4. Find the mathematical expectation of the number of the points obtained in a single throw of an unbiased die.

Answer:

Let X denote the no of points obtained by a single throw of an unbiased die.

Then probability distribution is as follow:

X	1	2	3	4	5	6
Prob.	1/6	1/6	1/6	1/6	1/6	1/6

$\therefore E(X) = 1.\frac{1}{6} + 2.\frac{1}{6} + 3.\frac{1}{6} + 4.\frac{1}{6} + 5.\frac{1}{6} + 6.\frac{1}{6} = \frac{6\times 7}{2\times 6} = \frac{7}{2} = 3.5$

5. The mean weight of 500 male students at a certain college is 150 lbs and the standard deviation is 15 lbs. Assuming that the weight is normally distributed find how many students weigh.

i) between 120 and 155 lbs

ii) more than 155 lbs.

[Given $\phi(2) = 0.9772;\ \phi(0.33) = 0.6293$ **]**

Answer:

Let X denote weight of a student.

Then $X \sim N(150, 225)$, So, $Z = \frac{X-150}{15} \sim N(0,1)$

Now, $P(120 \le X \le 155) = P\left(\frac{120-150}{15} \le \frac{X-150}{15} \le \frac{155-150}{15}\right)$

$= P(-2 \le Z \le 0.33) = (0.9772 - 0.5) + (0.6293 - 0.5)$

$= 0.4772 + 0.1293 = 0.6065$

$P(X > 155) = P\left(\frac{X-150}{15} > \frac{155-150}{15}\right) = P(Z > 0.33) = 1 - 0.6293 = 0.3707$

Hence the no. of students having weight between 120 lbs and 155 lbs

= 500×0.6065 = 303.25

Also the no. of students having weight more than 155 lbs

= 500×0.3707 = 185.35

6. A car hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with average number of demand per day 1.5. Calculate the proportion of days on which neither car is used and the proportion of days on which some demand is refused.

Answer:

Let X denotes the no of demands.

By hypothesis $X \sim P(1.5)$

$$\therefore P(X=0)=\frac{e^{-1.5}(1.5)^0}{0!}=e^{-1.5}$$

$$P(X>2)=1-P(X\le 2)=1-P(X=0)-P(X=1)-P(X=2)$$

$$=1-\frac{e^{-1.5}.(1.5)^0}{0!}-\frac{e^{-1.5}.(1.5)^1}{1!}-\frac{e^{-1.5}.(1.5)^2}{2!}$$

7. If a random variable *X* has Binomial distribution, find the mean and variance of the distribution with parameters (n, p).

Answer:

The m. g. f of the Binomial distribution $B(n_1, p)$ is given by $\varphi(t)=\left(q+pe^t\right)^n$

$\therefore$ Mean $=E(X)=\varphi'(0)=np$

$$E(X^2)=\varphi''(0)=n(n-1)p^2+np$$

$$\therefore \operatorname{var}(X)=E(X^2)-\{E(X)\}^2=n(n-1)p^2+np-n^2p^2=npq$$

8. Let X be a Poisson distributed random variable with the particular m; then show that E (X) = m and Var (X) = m.

Answer:

The m. g. f of the Poisson distribution $P_0(\mu)$ is

given by $\varphi(t)=e^{\mu(e^t-1)}$

Now Mean $=E(X)=\varphi'(0)=\mu e^0 e^{\mu(e^0-1)}$

$$E(X^2)=\varphi''(0)=\left(\mu e^0\right)^2 e^{\mu(e^0-1)}+\mu e^0 e^{\mu(e^0-1)}$$

$$\therefore \operatorname{var}(X)=E(X^2)-\{E(X)\}^2=\mu^2+\mu-\mu^2=\mu$$

9. If the chance of being killed by flood during a year is $1/3000$, use Poisson distribution to calculate probability that out of 3000 persons living in a village, at least one will die in flood in a year.

Answer:

Here $p=\frac{1}{3000},\ n=3000.$

$\therefore \lambda=np=1$

Hence $P(X\ge 1)=1-P(X=0)=1-\frac{e^{-1}1^0}{0!}=1-e^{-1}$

10. If *x* follows a Normal Distribution with mean 12 and variance 16, find $P(x\ge 20)$. [Given: $\int_{-\infty}^{2} 1/\sqrt{2\pi}\, e^{-1/2t^2}\,dt=0.977725$]

Answer:

Here $X \sim N(12,16)$

$$\therefore P(X\ge 20)=P\left(\frac{X-12}{4}\ge\frac{20-12}{4}\right)=P(Z\ge 2)=1-0.977725=0.022275$$

11. A random variable follows binomial distribution with mean 4 and standard deviation $\sqrt{2}$. Find the probability of assuming non-zero value of the variable.

Answer:
Let X be the said r.v.
Then $X \sim B(n, p)$ where $np = 4$ and $npq = 2$

Solving we get $q = \frac{1}{2}$, so $p = \frac{1}{2}$, $n = 8$

Hence $P(X > 0) = 1 - P(X = 0) = 1 - {}^{8}C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^8 = 1 - \left(\frac{1}{2}\right)^8$

12. If a random variable follows Poisson Distribution such that $P(1) = P(2)$.
Find (i) mean of the distribution
(ii) $P(4)$.

Answer:
Let X be the r.v. following Poisson distribution with $P(1) = P(2)$. Let λ be its parameter
Then $P(1) = P(2)$ gives

$$\frac{e^{-\lambda}.\lambda}{1!} = \frac{e^{-\lambda}.\lambda^2}{2!}$$

or, $\lambda^2 - 2\lambda = 0$
or, $\lambda(\lambda - 2) = 0$
$\therefore \lambda = 2$ since $\lambda = 0$ is not admissible

Now, $P(4) = \frac{e^{-\lambda}.\lambda^4}{4!} = \frac{e^{-2}.2^4}{4!} = \frac{e^{-2}.\cancel{16}}{\cancel{24}} = \frac{2}{3}e^{-2}$

Mean of the distribution $= \lambda = 2$

13. The probability density function of a continuous distribution is given by $f(x) = \frac{3}{4}x(2 - x)$. Compute mean and variance of the distribution.

Answer:
Let X denote the r.v. Its p.d.f. is $f(x) = \frac{3}{4}x(2 - x)$

Now, Mean $= E(X) = \int_{-\infty}^{\infty} xf(x)\,dx = \int_{-\infty}^{\infty} x.\frac{3}{4}x(2-x)\,dx = \frac{3}{4}\int_{-\infty}^{\infty} x^2(2-x)\,dx$

$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2.\frac{3}{4}x(2-x)\,dx = \frac{3}{4}\int_{-\infty}^{\infty} x^3(2-x)\,dx$

Hence $\mathrm{Var}(X) = E(X^2) - \{E(X)\}^2$

14. Find the mathematical expectation of the number of points obtained in a single throw of an unbiased die.
Answer:
Let X denote the number of points obtained in a single throw of an unbiased die. Then the probability table of X will be

X	1	2	3	4	5	6
Prob.	1/6	1/6	1/6	1/6	1/6	1/6

Hence $E(X)=1\times\frac{1}{6}+2\times\frac{1}{6}+3\times\frac{1}{6}+4\times\frac{1}{6}+5\times\frac{1}{6}+6\times\frac{1}{6}=3\frac{1}{2}$

15. Define Poisson distribution and find its mean and variance.

Answer:

The Poisson distribution with parameter λ is defined by the probability mass function $f(x)$ given by $f(x)=\frac{-e^{-\lambda}\lambda^x}{x!}$, $x=0,1,2,\cdots\cdots,n,n+1,\cdots\cdots$ to ∞.

The constant λ is called the parameter of the distribution.

Let X be a Poisson variate, i.e., $X \sim P(\lambda)$

Then Mean $=E(X)=\sum_{x=0}^{\infty}x\frac{e^{-\lambda}\lambda^x}{x!}=e^{-\lambda}\left\{\frac{1\cdot\lambda}{1!}+\frac{2\cdot\lambda^2}{2!}+\frac{3\cdot\lambda^3}{3!}+\cdots\cdots\right\}$

$$=\lambda e^{-\lambda}\left\{1+\frac{\lambda}{1!}+\frac{\lambda^2}{2!}+\cdots\cdots\right\}=\lambda e^{-\lambda}\cdot e^{\lambda}=\lambda.$$

Again, $E(X^2)=\sum_{x=0}^{\infty}x^2\frac{e^{-\lambda}\lambda^x}{x!}=e^{-\lambda}\sum_{x=0}^{\infty}\{x(x-1)+x\}\frac{\lambda^x}{x!}$

$$=e^{-\lambda}\sum_{x=2}^{\infty}\frac{\lambda^x}{(x-2)!}+e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^x}{(x-1)!}=e^{-\lambda}\cdot\lambda^2\cdot e^{\lambda}+e^{-\lambda}\cdot\lambda\cdot e^{\lambda}$$

$$=\lambda^2+\lambda$$

Hence, $\operatorname{var}(X)=E(X^2)-\{E(X)\}^2=\lambda^2+\lambda-\lambda^2=\lambda$.

16. Suppose that an airplane engine will fail, when in flight, with probability $(1-p)$ independently from engine to engine; suppose that the airplane will make a successful flight if at least 50% of its engines remain operative. For what values of *p* is a four-engine plane preferable to a two-engine plane?

Answer:

A four-engine plane is preferable to a two-engine plane if 4-engine plane flies more successfully than a 2-engine plane i.e., if two or more engines of a 4-engine plane remain operative

P (1 or more engines of a 2-engine plane remain operative)

But, P (2 or more engines remains operative)

$$={}^4C_2p^2(1-p)^2+{}^4C_3p^3(1-p)+{}^4C_4p^4$$

P (1 or more engines remains operative) $={}^2C_1p(1-p)+{}^2C_2p^2$

Hence the required condition is

$${}^4C_2p^2(1-p)^2+{}^4C_3p^3(1-p)+{}^4C_4p^4>{}^2C_1p(1-p)+{}^2C_2p^2$$

or, $6p^2(1-p)^2+4p^3(1-p)+p^4-2p(1-p)-p^2>0$

or, $6p^2-12p^3+6p^4+4p^3-4p^4+p^4-2p+2p^2-p^2>0$

or, $3p^4-8p^3+7p^2-2p>0$

or, $3p^3-8p^2+7p-2>0$ [As $p>0$ by definition]

$3p^2-5p+2>0$ as $p\neq 1$

$$p^2-\frac{5}{3}p+\frac{2}{3}>0$$

$$\left(p-\frac{5}{6}\right)^2>\frac{25}{36}-\frac{2}{3}=\frac{1}{36}$$

or, $\quad p > \frac{5}{6} + \frac{1}{6}$

or, $\quad p < \frac{5}{6} - \frac{1}{6} \qquad$ or, $< \frac{2}{3}$

17. Determine the mean and variance of exponential distribution.

Answer:

The exponential distribution $E(\mu)$ is given by the p.d.f.

$$f(x) = \begin{cases} \mu e^{-\mu x} & \text{if} \quad x \ge 0 \\ 0 & \text{if} \quad x < 0 \end{cases}, \qquad \mu > 0$$

Now $\quad \text{Mean} = E(X) = \int_0^\infty x\mu e^{-\mu x} dx$

Putting $\mu x = y$, $\mu dx = dy$

$$\therefore \qquad E(X) = \int_0^\infty y e^{-y} \frac{dy}{\mu} = \frac{1}{\mu}\int_0^\infty y d\left(-e^{-y}\right) = \frac{1}{\mu}\left[-ye^{-y}\right]_0^\infty = \frac{1}{\mu}$$

Similarly, $E\left(X^2\right) = \int_0^\infty x^2 \mu e^{-\mu x} dx = \frac{2}{\mu^2}$

Hence, $\operatorname{var}(X) = E\left(X^2\right) - \left\{E(X)\right\}^2 = \frac{2}{\mu^2} - \frac{1}{\mu^2} = \frac{1}{\mu^2}$.

18. Find the mean of an uniform distribution.

Answer:

The uniform distribution $U(a,b)$ of a.r.v. X is given by the p.d.f.

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Mean} = \int_{-\infty}^{\infty} f(x) dx = \int_a^b \frac{1}{b-a} dx = \frac{a+b}{2}$$

19. Define Normal distribution and find its mean, variance and standard deviation.

Answer:

A random variable X is said to have normal distribution if its probability density function is given by $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$, $\quad -\infty < x < +\infty$.

Where m, $\sigma(>0)$ are two parameters of this distribution. If any random variable have normal distribution with parameters m and σ then we write $X \sim N(m,\sigma)$.

Mean: $E(X) = \int_{-\infty}^{\infty} xf(x) dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} xe^{-\frac{(x-m)^2}{2\sigma^2}} dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-m) e^{-\frac{(x-m)^2}{2\sigma^2}} dx + \frac{m}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-m)^2}{2\sigma^2}} dx$$

Now we substitute $\dfrac{x-m}{\sigma\sqrt{2}} = z$ for the first integral;

$$= \sigma\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} z e^{-z^2}\,dx + m\int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx$$

$$= 0 + m$$

$$= m$$

Variance: $E(X-m)^2 = \displaystyle\int_{-\infty}^{\infty}(x-m)^2 f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-m)^2 e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx$

Substitute: $\dfrac{x-m}{\sigma\sqrt{2}} = z$;

$$= \frac{2\sigma^2}{\sqrt{\pi}}\int_{-\infty}^{\infty} z^2 e^{-z^2}\,dx = \frac{4\sigma^2}{\sqrt{\pi}}\int_{0}^{\infty} z^2 e^{-z^2}\,dx$$

Subs: $z^2 = t \Rightarrow 2z\,dz = dt$

$$= \frac{2\sigma^2}{\sqrt{\pi}}\int_{0}^{\infty} t^{\frac{1}{2}} e^{-t}\,dx = \frac{2\sigma^2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right) = \frac{2\sigma^2}{\sqrt{\pi}}\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \sigma^2.$$

Standard deviation $= \sqrt{\text{Variance}} = \sigma$.

20. A normal population has mean 0.1 and standard deviation 2.1. Find the probability that the mean of a sample of size 900 will be negative. [Given that $P(|z| < 1.43 = 0.847)$]

Answer:

Here $X \square N(0, 1, 2.1^2),\ n = 900$. That is, $\mu = 0.1,\ \sigma = 2.1$

The required probability $= P(\bar{X} < 0)$

$$= P\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < \frac{-\mu}{\sigma/\sqrt{n}}\right) = P\left(Z < \frac{-0.1}{2.1/30}\right)$$

$$= P(Z < -1.43) = \frac{1}{2}(1-0.847) = -0.0765$$

21. The mean yield for one acre plot is 662 kilos with a standard deviation 32 kilos. Assuming normal distribution how many one acre plots in a batch of 1000 plots would you expect to have yield over 700 kilos? (Given that $\phi(1.19) = 0.3830$)

Answer:

Here, $\mu = 662$ kilos, $\sigma = 32$ kilos.

Let X denote yield of an one acre plot.

Then $P(X > 700 \text{ kilos}) = P\left(\dfrac{X-\mu}{\sigma} > \dfrac{700-662}{32}\right)$

$$= P\left(Z > \frac{38}{32}\right), \text{denoting } Z = \frac{X-\mu}{\sigma}$$

$$= P(Z > 1.19) = 0.5 - 0.3850 = 0.115$$

Hence no of plots yielding more than 700 kilos $= 1000 \times 0.115 = 115$

21. Let X be uniformly distributed over the interval $([1,\ 2]$ and $\bar{X}=E(X))$. Find out the value of a so that $P(X>a+\bar{X})=\frac{1}{6}$.

Answer:

Given $X \sim \cup(1,2)$. Then its $p-d-f$ $f(x)$ is given by

$$f(x)=\begin{cases}1 \text{ if } 1<x<2 \\ 0 \text{ if } \text{ otherwise}\end{cases}$$

$$E(X)=\int_{-\infty}^{\infty} x\,f(x)dx=\int_1^2 x\,dx=\left[\frac{x^2}{2}\right]_1^2=\frac{3}{2}$$

Now, $P\left(X>a+\frac{3}{2}\right)=\frac{1}{6}$

or, $\int_{a+3/2}^{\infty} f(x)dx=\frac{1}{6}$

or, $\int_{a+3/2}^{2} 1.dx=\frac{1}{6}$

or, $2-a-3/2=\frac{1}{6}$

or, $a=\frac{1}{2}-\frac{1}{6}=\frac{1}{3}$

23. Examine whether the function $|x|$ in (–1, 1) and zero elsewhere is a density function.

Answer:

The given function is

$$f(x)=\begin{cases}|x| & \text{for } x\in(-1,1) \\ 0 & \text{elsewhere}\end{cases}$$

Clearly $f(x)\geq 0$ on $\mathbb{R}$ and $\int_{-\infty}^{\infty} f(x)dx=\int_{-1}^{0}(-x)dx+\int_0^1 x\,dx=1$

Hence $f(x)$ is a density function as $f(x)\geq 0$ and $\int_{-\infty}^{\infty} f(x)dx=1$.

24. In a certain city, the daily consumption of electric power (in millions of kilowatt hours) is a random variable having the probability density

$$f(x)=\frac{1}{9}xe^{-x/3},\ x>0$$
$$=0,\qquad x\leq 0$$

If the city's power plant has a daily capacity of 12 million kilowatt-hours, what is the probability that this power supply will be inadequate on any given day?

Answer:

Here Prob. (Power supply is inadequate on a day) $=P(X>12)=\frac{1}{9}\int_{12}^{\infty} xe^{-\frac{x}{3}}dx$

25. The probability density of a random variable *z* is given by

$$f(z)=\begin{cases} kze^{-z^2}, & \text{for } z>0 \\ 0 & \text{for } z\le 0 \end{cases}$$

Find the value of *k* and find out the corresponding distribution function of *z*.

Answer:

As $f(z)$ is a p.d.f. of a.r.v, we have $f(z)\ge 0$ and $\int_{-\infty}^{\infty} f(z)\,dz=1$

Thus $\int_0^{\infty} kze^{-z^2}\,dz=1$

Putting $t=z^2,\ 2z\,dz=dt$, when $z=0, t=0$ and when $z\to\infty, t\to\infty$.

$\therefore\qquad \int_0^{\infty} e^{-t}\,dt=\frac{2}{k}$

or, $\qquad \left[-e^{t}\right]_0^{\infty}=\frac{2}{k}$

or, $\qquad l=\frac{2}{k}$

or, $\qquad k=2$

The distribution function $F(z)$ is given by

$$F(z)=\int_{-\infty}^{z} f(x)\,dx$$

$$=\begin{cases} \int_0^{\infty} 2xe^{-x^2} & \text{if} \quad z>0 \\ 0 & \text{if} \quad z\le 0 \end{cases}$$

$$=\left[-e^{-t}\right]_0^{z} \qquad \text{if} \qquad z>0, =0 \text{ if } z\le 0 = \begin{cases} 1-e^{-z} & \text{if} \quad z>0 \\ 0 & \text{if} \quad z\le 0 \end{cases}$$

26. If X_1, X_2, X_3, ……, X_n constitute a random sample of size n from an infinite population with mean μ and variance σ^2, then prove that $E(\overline{X})=\mu$ and $\text{var}(\overline{X})=\frac{\sigma^2}{n}$.

Answer:

We see $\overline{X}=\frac{1}{n}(X_1+X_2+........+X_n)$

$\therefore E(\overline{X})=\frac{1}{n}\{E(X_1)+E(X_2)+...+E(x_n\}=\frac{1}{n}\{\mu+\mu+...+\mu\}=\frac{1}{n}\cdot n\mu=\mu$

$Var(\overline{X})=\frac{1}{n^2}\{Var(X_1)+Var(X_2)+...+Var(X_n)\}$

$=\frac{1}{n^2}\{\sigma^2+\sigma^2+.....+\sigma^2\}=\frac{n\sigma^2}{n^2}=\frac{\sigma^2}{n}$

Recall $Var(X)=E(x-\overline{X})^2$

27. The life of a tyre manufactured by a company follows a continuous distribution given by the density function

$$f(x)=\frac{k}{x^3}, \qquad 1000\le x\le 1500$$

$= 0,$ **elsewhere**

Find the value of *k* and find the probability that a randomly selected tyre would function for at least 1200 hours.

Answer:

Since $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{1000}^{1500} \frac{k}{x^2}\,dx = 1$$

or, $\int_{1000}^{1500} x^{-2}\,dx = \frac{1}{k}$

or, $\left[\frac{-1}{x}\right]_{1000}^{1500} = \frac{1}{k}$

or, $\frac{1}{1000} - \frac{1}{1500} = \frac{1}{k}$

$\therefore k = 3000$

$$\therefore P(x \geq 1200\,\text{hrs}) = \int_{1200}^{1500} \frac{3000}{x^2}\,dx = 3000\left[-\frac{1}{x}\right]_{1200}^{1500} = 3000\left(\frac{1}{1200} - \frac{1}{1500}\right) = \frac{1}{2}$$

28. If the r.v X takes the values 1, 2, 3 and 4 such that 2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4), find the probability distribution.

Answer:

Since $P(X=1) + P(X=2) + P(X=3) + P(X=4) = 1$

$$P(X=1) + \frac{2}{3}P(X=1) + 2P(X=3) + \frac{2}{5}P(X=1) = 1$$

$$P(X=1) = \frac{15}{61}$$

$$\therefore P(X=2) = \frac{2}{3} \times \frac{15}{61} = \frac{10}{61}$$

$$P(X=3) = 2 \times \frac{15}{61} = \frac{30}{61}$$

$$P(X=4) = \frac{2}{5} \times \frac{15}{61} = \frac{6}{61}.$$

29. If the weekly wage of 10,000 workers in a factory follows normal distribution with mean and standard deviation Rs. 70 and Rs. 5 respectively, then find the expected number of workers whose weekly wages are

i) betweenRs. 66 and Rs. 72

ii) less than Rs. 66.

[Given that the area under the standard normal curve between *z* = 0 and *z* = 0.4 is 0.1554 and *z* = 0 and *z* = 0.8 is 0.2881].

Answer:

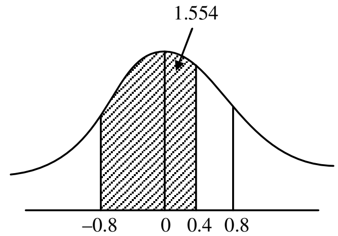
Let X denote weekly wage of a worker.

Then $X \sim N(70, 25)$

Now $P(66 < X < 72) = P\left(\frac{66-70}{5} < \frac{X-70}{5} < \frac{72-70}{5}\right)$

$$= P(-0.8 < Z < 0.4) = 0.6554 - (1 - 0.7881)$$

$$= 0.6554 - 0.2119 = 0.4435$$
$$= 0.1554 + 0.2881$$
$$P(X < 66) = P\left(\frac{X-70}{5} < \frac{66-70}{5}\right) = P(Z < -0.8) = 0.2119$$



Hence the expected no. of workers with weekly wage

(a) between Rs.66 and Rs.72 $= 0.4435 \times 10,000 = 4435$

(b) less than Rs.60 $= 0.2119 \times 10,000 = 2119$.

30. If the weekly wages of 10,000 workers in a family follow normal distribution with mean and st-dev. Rs. 70 and Rs. 5 respectively. Find the expected no. of workers whose weekly wages are (i) between Rs. 66 and Rs. 72 (ii) less than Rs. 66 (iii) mere than Rs. 72.

Answer:

Let X denote wage of a worker

Then $X \sim N(70,25)$

Now, $P(66 \le X \le 72) = P\left(\frac{66-70}{5} \le \frac{X-70}{5} \le \frac{72-70}{5}\right)$

$$= P(-0.8 \le Z \le 0.4), \text{writing } Z = \frac{X-70}{5}$$

$$P(X < 66) = P\left(\frac{X-70}{5} < \frac{66-70}{5}\right) = P(Z < -08)$$

$$P(X > 72) = P\left(\frac{X-70}{5} > \frac{72-70}{5}\right) = P(Z > 0.4)$$

Hence the expected no. of workers whose monthly ways lies between Rs.66 and Rs.72

$= 10,000 \times P(66 \le X \le 72)$

The expected no. of workers with monthly wage less than Rs.66

$= 10,000 \times P(X < 66)$

The expected no. of workers with monthly wage more than Rs.72

$= 10,000 \times P(X > 72)$

31. If the probability density function of a random variable X is given by $f(x) = c.e^{-(x-2)^2/2}$ $-\infty < x < \infty$, find the value of c, the expectation and variance of the distribution.

Answer:

As $\int_{-\infty}^{\infty} f(x)\,dx = 1$, we have

$$C\int_{-\infty}^{\infty} e^{-\frac{(x-2)^2}{2}}\,dx = 1$$

Now putting $\frac{x-2}{\sqrt{2}} = z$

$dx = \sqrt{2}dz$ and when $x \to -\infty, z \to \infty$ and when $x \to \infty, z \to \infty$.

Thus $C\int_{-\infty}^{\infty} e^{-z^2}\sqrt{2}dz = 1$ or, $C\int_{-\infty}^{\infty} e^{-z^2}dz = \frac{1}{\sqrt{2}C}$